

Pair correlation of roots of rational functions with rational generating functions and quadratic denominators

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Dedicated in honor of Mourad Ismail and Dennis Stanton

Abstract

For any rational functions with complex coefficients $A(z)$, $B(z)$ and $C(z)$, where $A(z)$, $C(z)$ are not identically zero, we consider the sequence of rational functions $H_m(z)$ with generating function $\sum H_m(z)t^m = 1/(A(z)t^2 + B(z)t + C(z))$. We provide an explicit formula for the limiting pair correlation function of the roots of $\prod_{m=0}^n H_m(z)$, as $n \rightarrow \infty$, counting multiplicities, on certain closed subarcs J of a curve \mathcal{C} where the roots lie. We give an example where the limiting pair correlation function does not exist if J contains the endpoints of \mathcal{C} .

1 Introduction

The root distribution of certain sequences of rational functions with complex coefficients has some peculiar connections with the discriminant of the denominator of its generating function. In particular, one of the authors [22] showed that if the generating function of a sequence of polynomials is the reciprocal of a quadratic or cubic polynomial, then the roots of the corresponding sequence of polynomials form a dense set on an explicit algebraic curve. Moreover, in several cases, the endpoints of the curve are the roots of the discriminant of the denominator. For the root distribution of various sequences of polynomials and their generating functions, see Figure 1, Figure 2 and Figure 3 below. For other connections between discriminants and the root distribution, see [23]. Our goal here is to study the pair correlation of the roots of a sequence of rational functions when the denominator of the generating function is quadratic. Pair correlation is one of the important statistics concerned with the local spacing distribution of a sequence, and there are many results

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in the literature on the local spacing distribution of a variety of sequences of interest in number theory (see [1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21]).

Let \mathcal{T} denote the set of triples $(A(z), B(z), C(z))$ with $A(z), B(z), C(z)$ rational functions of z with complex coefficients, $A(z)$ and $C(z)$ not identically zero. For each $(A(z), B(z), C(z))$ in \mathcal{T} we consider the sequence of rational functions $H_m(z)$ with generating function given by

$$\sum_{m=0}^{\infty} H_m(z) t^m = 1/(A(z)t^2 + B(z)t + C(z)). \quad (1)$$

It is easy to see that there exists a finite set of points in the complex plane which contains all the poles of all the rational functions $H_m(z)$, $m = 0, 1, \dots$ (the multiplicity of the poles may increase as m increases). By contrast, if we consider all the zeros of all $H_m(z)$, we obtain an infinite set of complex numbers which (except in some degenerate cases) is dense on a certain curve \mathcal{C} which will be described below. This naturally raises the question of how these points are distributed along \mathcal{C} .

Our goal is to study the pair correlation of the roots of $\prod_{m=0}^n H_m(z)$, counting multiplicities. The rational function $B^2(z)/A(z)C(z)$ plays an important role in our investigation. In this connection, let us consider the equivalence relation \sim on \mathcal{T} defined such that two elements of \mathcal{T} are equivalent if and only if the corresponding rational function $B^2(z)/A(z)C(z)$ is the same. We will say that an element $(A(z), B(z), C(z))$ of \mathcal{T} is in standard form provided $C(z) = 1$, $A(z)$ and $B(z)$ are polynomials, and there is no root of $B(z)$ which is simultaneously also a double (or higher order) root of $A(z)$. The last condition above can be restated as saying that the greatest common divisor $(B^2(z), A(z))$ of $B^2(z)$ and $A(z)$ is square free. As examples, the triples $(A(z), B(z), C(z))$ used in Figures 1 and 2 below are in standard form, while the one used in Figure 3 is not in standard form, since in that example $(B^2(z), A(z)) = z^2$. It is easy to see that in every equivalence class there is exactly one element of \mathcal{T} in standard form. Indeed, notice first that for any $(A(z), B(z), C(z))$ in \mathcal{T} , $(A(z), B(z), C(z))$ and $(A(z)/C(z), B(z)/C(z), 1)$ belong to the same equivalence class. Also, for any nonzero rational function $E(z)$, the triple $(E^2(z)A(z)/C(z), E(z)B(z)/C(z), 1)$ belongs to the same equivalence class. Here one can choose $E(z)$ such that $(E^2(z)A(z)/C(z), E(z)B(z)/C(z), 1)$ is in standard form. For instance, choose first a polynomial $E(z)$ such that both $E^2(z)A(z)/C(z)$ and $E(z)B(z)/C(z)$ are polynomials. If $E(z)B(z)/C(z)$ has a root α which is also a multiple root of $E^2(z)A(z)/C(z)$, divide $E(z)$ by $z - \alpha$. Then continue this procedure until all such roots are eliminated. It follows that any equivalence class contains at least one triple in standard form. Next, if $(A_1(z), B_1(z), C_1(z)) \sim (A_2(z), B_2(z), C_2(z))$ and both triples are in standard form, then $C_1(z) = C_2(z) = 1$ and $B_1^2(z)/A_1(z) = B_2^2(z)/A_2(z)$. Therefore $B_1^2(z)$ divides $B_2^2(z)A_1(z)$, and since $(B^2(z), A(z))$ is square free, this forces $B_1(z)$ to divide $B_2(z)$. Similarly, $B_2(z)$ divides $B_1(z)$, so $B_1(z) = B_2(z)$ and then $A_1(z) = A_2(z)$. In conclusion, each equivalence class contains exactly one triple in standard form. Let us also remark that, given an arbitrary triple $(A(z), B(z), C(z))$ in \mathcal{T} , the process of finding the unique triple in its equivalence class which is in standard form described above also provides us with a clear understanding of how the roots of the corresponding rational functions $H_m(z)$ change. More precisely, replacing $(A(z), B(z), C(z))$ by $(A(z)/C(z), B(z)/C(z), 1)$ has the effect of multiplying each $H_m(z)$ by the fixed rational function $C(z)$. Moreover, replacing $(A(z)/C(z), B(z)/C(z), 1)$ by $(E^2(z)A(z)/C(z), E(z)B(z)/C(z), 1)$ has the same effect as replacing t by $E(z)t$, that is, has the effect of multiplying each $H_m(z)$ by $E(z)^m$. In conclusion, the distribution of zeros of the rational functions $H_m(z)$ corresponding to a given triple $(A(z), B(z), C(z))$ in \mathcal{T} is exactly the same as the one obtained by replacing $(A(z), B(z), C(z))$ by the unique triple in standard form in its equivalence class, except at the location of finitely many points in the complex plane (where the zeros and poles of $C(z)$ and $E(z)$ above lie).

Taking into account the above discussion, we restrict ourselves in what follows to study the pair correlation of zeros of $\prod_{k=0}^m H_k(z)$ for triples $(A(z), B(z), C(z))$ which are in standard form. Note that in this case all the $H_m(z)$ are polynomials, and $H_0(z) = 1$, so we may restrict the above product to $k \geq 1$. We consider the pair correlation problem on an arbitrary subarc J of the curve \mathcal{C} on which the roots of $H_m(z)$ lie. This restriction sometimes implies an exclusion of the endpoints of \mathcal{C} from J . We will see an explicit example in Section 3 where the limiting pair correlation function exists on any proper subinterval which does not contain these endpoints. This function does not exist if these endpoints belong to J . The pair correlation function is explicitly given in the following theorem.

Theorem 1 *Let $A(z)$ and $B(z)$ be polynomials in z with complex coefficients, $A(z)$ not identically zero, such that the greatest common divisor of $B^2(z)$ and $A(z)$ is square free, and consider the polynomials $H_m(z)$ with generating function*

$$\sum_{m=0}^{\infty} H_m(z) t^m = \frac{1}{A(z)t^2 + B(z)t + 1}.$$

Then the set of roots of all the $H_m(z)$ is dense on a fixed curve \mathcal{C} . Also, let

$$h(z) = \frac{B^2(z)}{A(z)}$$

and let J be a closed subarc of \mathcal{C} such that $h(z)$ is piecewise continuously differentiable on J and the inverse function exists. The function

$$f(t) := h^{-1}(4 \cos^2 \pi t)$$

maps a subinterval $I \subset [0, 4]$ onto J . If $f'(z) \neq 0$ on I then the limiting pair correlation function of the roots of $\prod_{k=1}^m H_k(z)$, counting multiplicities, exists on J , as $m \rightarrow \infty$, and is given by

$$g_J(x) = \frac{l(J)}{|I|^2} \int_I g_I \left(\frac{l(J)}{|I|} x \right) \frac{dt}{|f'(t)|}, \quad (2)$$

where

$$g_I(x) = \frac{6}{\pi^2 x^2} \sum_{1 \leq k \leq 2x} \sigma(k) \log \frac{2x}{k} \quad (3)$$

and σ is the sum of divisors function.

The first part of the theorem, stating that the set of roots of all $H_m(z)$ is dense on a fixed curve \mathcal{C} has been established in [22]. For the sake of completeness, we will present a proof in Section 2. The proof of the main part of the theorem involving the study of pair correlation is given in Section 3. The methodology of the proof is similar to that used in [2], and with the one employed in [1], where a more elaborate argument is needed since it addresses the effect of addition on pair correlation functions concerning fractions of bounded height. Section 4 provides an example of the pair correlation function for a specific sequence of polynomials.

2 Distribution of roots on a fixed curve

In this section we consider the root distribution of the sequence of polynomials $H_m(z)$. We recall the q -analogue of the discriminant, a very useful concept introduced by Mourad Ismail [13]. The q -discriminant of a polynomial $P(x)$ of degree n with the leading coefficient p is

$$\text{Disc}_x(P(x); q) = p^{2n-2} q^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (q^{-1/2} x_i - q^{1/2} x_j)(q^{1/2} x_i - q^{-1/2} x_j)$$

where x_1, \dots, x_n are roots of $P(x)$. This q -discriminant equals 0 if and only $x_i/x_j = q$ for some roots x_i, x_j . In the special case when $q \rightarrow 1$, this q -discriminant gives the ordinary discriminant of a polynomial. The following theorem was established in [22].

Theorem 2 *Let*

$$\frac{1}{A(z)t^2 + B(z)t + 1} = \sum H_m(z)t^m,$$

where $A(z) \neq 0$. The roots of $H_m(z)$ are dense on the curve \mathcal{C} defined by the conditions

$$\Im \frac{B^2(z)}{A(z)} = 0$$

and

$$0 \leq \Re \frac{B^2(z)}{A(z)} \leq 4$$

in the complex plane.

Proof Let z be a root of $H_m(z)$ with $A(z) \neq 0$. Let $t_1 = t_1(z)$ and $t_2 = t_2(z)$ be the roots of $A(z)t^2 + B(z)t + 1$. If $t_1 = t_2$ then $z \in \mathcal{C}$ since $B^2(z) = 4A(z)$. We consider $t_1 \neq t_2$. Using partial fractions, we have

$$\begin{aligned} \frac{1}{A(z)t^2 + B(z)t + 1} &= \frac{1}{A(z)(t - t_1)(t - t_2)} \\ &= \frac{1}{A(z)} \sum \frac{t_1^{m+1} - t_2^{m+1}}{(t_1 - t_2)t_1^{m+1}t_2^{m+1}} t^m. \end{aligned} \quad (4)$$

So the roots of $H_m(z)$ are the roots of $t_1 = qt_2$, where q is an $(m+1)$ -th root of unity and $q \neq 1$, $A(z) \neq 0$. These roots are the roots of the q -analogue of discriminant

$$\text{Disc}_t(A(z)t^2 + B(z)t + 1; q) = q(B^2(z) - (q + q^{-1} + 2)A(z)).$$

Hence

$$\begin{aligned} \frac{B^2(z)}{A(z)} &= q + q^{-1} + 2 \\ &= 2\Re q + 2 \end{aligned}$$

Thus $z \in \mathcal{C}$ since q is an $(m+1)$ -th root of unity.

To show the density of the roots of $H_m(z)$, we let $\zeta \in \mathcal{C}$ and U be an open neighborhood of ζ such that the rational function $B^2(z)/A(z)$ is analytic on U . Since the set of $(m+1)$ -th roots of

unity is dense on the unit circle as $m \rightarrow \infty$, the set of values $2\Re q + 2$ is dense on the interval $[0, 4]$ where q is an $(m + 1)$ -th root of unity. By the open mapping theorem, the map $B^2(z)/A(z)$ maps U to an open set containing some point $2\Re q + 2$ for large m . Thus there is some point $z \in U$ such that

$$\frac{B^2(z)}{A(z)} = q + q^{-1} + 2$$

or

$$\text{Disc}_t(A(z)t^2 + B(z)t + 1; q) = 0.$$

This implies that q is the quotient of the two roots of $A(z)t^2 + B(z)t + 1$. Hence z is a root of $H_m(z)$ by (4).

The root distribution of various sequences of polynomials with their generating functions are given by figures below.

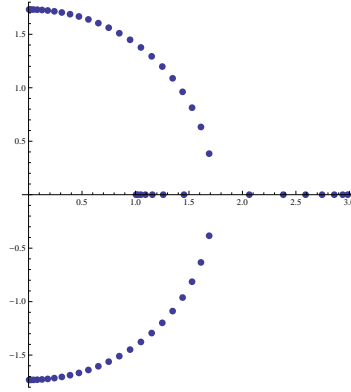


Figure 1: Roots of $H_{30}(z)$ with $\sum H_m(z)t^m = 1/(z^2t^2 + (z^2 - 2z + 3)t + 1)$

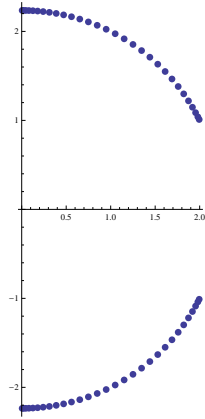


Figure 2: Roots of $H_{30}(z)$ with $\sum H_m(z)t^m = 1/(z^2t^2 + (z^2 - 2z + 5)t + 1)$

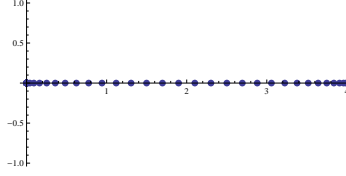


Figure 3: Roots of $H_{30}(z)$ with $\sum H_m(z)t^m = 1/(z^2t^2 + (z^2 - 2z)t + 1)$

In the next section, we will analyze the pair correlation of the roots of $\prod_{m=1}^n H_m(z)$, counting multiplicities.

3 Pair correlation on the corresponding curve

In this section, we will prove Theorem 1. In the proof of Theorem 2, the map $h(z)$ maps a root of $H_m(z)$ to $q + q^{-1} + 2$ where q is a $(m+1)$ -th root of unity. So it is essential to consider the pair correlation function of the sequence of finite sequences $M(Q) = \{x_1, \dots, x_N\} = \{1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, 2/5, 3/5, 4/5, \dots\}$ on a subinterval $I = [a, b]$ of $(0, 1)$. Each finite sequence is a sequence of $N = Q(Q-1)/2$ numbers p/q with $1 \leq p < q$ and $q \leq Q$. Let $M_I(Q) = M(Q) \cap I$. The number of points $N_I(Q)$ in the interval I is

$$\begin{aligned} N_I(Q) &= \sum_{q \leq Q} \sum_{qa \leq p \leq qb} 1 \\ &= \sum_{q \leq Q} q(b-a) + O(1) \\ &= \frac{(b-a)Q^2}{2} + O(Q) \\ &= N(b-a) + O(Q). \end{aligned}$$

We consider the quantity

$$R_{M_I(Q)}(\lambda) = \frac{1}{2N(b-a)} \# \left\{ (x_i, x_j) \in M_I(Q) : i \neq j, 0 < |x_i - x_j| \leq \frac{\lambda}{N} \right\}.$$

Let the limiting pair correlation measure $R_{M_I}(\lambda) = \lim_{Q \rightarrow \infty} R_{M_I(Q)}(\lambda)$ if the limit exists. If moreover $R_{M_I(Q)}(\lambda)$ can be written in the form

$$R_{M_I}(\lambda) = \int_0^\lambda g_I(x) dx,$$

the function $g_I(\lambda)$ is called the pair correlation function associated to the given sequence of finite sets $(M_I(Q))_{Q \in \mathbb{N}}$. We will need the following lemma. The proof of this lemma is similar to that of Lemma 1 in [2].

Lemma 3 *For any subinterval $I \subset (0, 1)$, the pair correlation function of the sequence of finite sets $(M_I(Q))_{Q \in \mathbb{N}}$ exists and is given by*

$$g_I(x) = \frac{6}{\pi^2 x^2} \sum_{1 \leq k \leq 2x} \sigma(k) \log \frac{2x}{k}$$

for any $x > 0$.

Proof Let G and H be smooth functions defined on \mathbb{R} with $\text{Supp } G \subset (0, 1)$ and $\text{Supp } H \subset (0, \Lambda)$. Here $\text{Supp } G$ denotes the closure in \mathbb{R} of the set of points where G is nonzero, and similarly for $\text{Supp } H$. Let

$$\begin{aligned} h(y) &= \sum_{n \in \mathbb{Z}} H(N(y + n)), \\ g(y) &= \sum_{n \in \mathbb{Z}} G(y + n) \end{aligned}$$

and

$$S = \sum_{x, y \in M(Q)} h(x - y) g(x) g(y).$$

We note that the summations in the definitions of $h(y)$ and $g(y)$ contain only finitely many nonzero terms. This allows us to interchange summations in the subsequent computations. The functions $h(y)$, $g(y)$ have Fourier series expansions,

$$h(y) = \sum_{n \in \mathbb{Z}} c_n e(ny)$$

and

$$g(y) = \sum_{n \in \mathbb{Z}} a_n e(ny).$$

By substituting these Fourier expansions into the formula of S , we obtain

$$\begin{aligned} S &= \sum_{x, y \in M(Q)} \sum_m c_m e(m(x - y)) \sum_n a_n e(nx) \sum_r a_r e(ry) \\ &= \sum_{m, n, r} c_m a_n a_r \sum_{x \in M(Q)} e((m + n)x) \sum_{y \in M(Q)} e((r - m)y) \\ &= \sum_{m, n, r} c_m a_n a_r \sum_{q \leq Q} \sum_{1 \leq p < q} e\left(\frac{(m + n)p}{q}\right) \sum_{q \leq Q} \sum_{1 \leq p < q} e\left(\frac{(r - m)p}{q}\right) \\ &= \sum_{m, n, r} c_m a_n a_r \left(\sum_{\substack{q \leq Q \\ q \nmid m+n}} \sum_{1 \leq p < q} e\left(\frac{(m + n)p}{q}\right) + \sum_{\substack{q \leq Q \\ q \nmid m+n}} \sum_{1 \leq p < q} e\left(\frac{(m + n)p}{q}\right) \right) \\ &\quad \times \left(\sum_{\substack{q \leq Q \\ q \nmid r-m}} \sum_{1 \leq p < q} e\left(\frac{(r - m)p}{q}\right) + \sum_{\substack{q \leq Q \\ q \nmid r-m}} \sum_{1 \leq p < q} e\left(\frac{(r - m)p}{q}\right) \right). \end{aligned}$$

Using the fact that for any integer l ,

$$\sum_{1 \leq p \leq q} e\left(\frac{lp}{q}\right) = \begin{cases} 0 & \text{if } q \nmid p, \\ q & \text{if } q|p, \end{cases}$$

we obtain

$$\begin{aligned} S &= \sum_{m,n,r} c_m a_n a_r \left(\sum_{\substack{q \leq Q \\ q|m+n}} (q-1) - \sum_{\substack{q \leq Q \\ q \nmid m+n}} 1 \right) \left(\sum_{\substack{q \leq Q \\ q|r-m}} (q-1) - \sum_{\substack{q \leq Q \\ q \nmid r-m}} 1 \right) \\ &= \sum_{m,n,r} c_m a_n a_r \left(\sum_{\substack{q \leq Q \\ q|m+n}} q - Q \right) \left(\sum_{\substack{q \leq Q \\ q|r-m}} q - Q \right) \\ &= \sum_{m,n,r} c_m a_n a_r \sum_{\substack{q \leq Q \\ q|m+n}} q \sum_{\substack{q \leq Q \\ q|r-m}} q - Q \sum_{m,n,r} c_m a_n a_r \sum_{\substack{q \leq Q \\ q|m+n}} q \\ &\quad - Q \sum_{m,n,r} c_m a_n a_r \sum_{\substack{q \leq Q \\ q|r-m}} q + Q^2 \sum_{m,n,r} c_m a_n a_r. \end{aligned}$$

By the Poisson summation formula (see [7, Theorem 2.2, page 37] or [10] for a generalized version), we have

$$\sum_{r \in \mathbb{Z}} a_r = \sum_{r \in \mathbb{Z}} g(r) = 0$$

since $\text{Supp} G \subset (0, 1)$. With this fact, S is reduced to

$$S = \sum_{m,n,r} c_m a_n a_r \sum_{\substack{q \leq Q \\ q|m+n}} q \sum_{\substack{q \leq Q \\ q|r-m}} q.$$

With the change in the index of summation given by $m' = m + n$, $n' = r - m$ and $m = r'$, we obtain

$$\begin{aligned} S &= \sum_{m',n',r'} c_{r'} a_{m'-r'} a_{n'+r'} \sum_{\substack{q \leq Q \\ q|m'}} q \sum_{\substack{q \leq Q \\ q|n'}} q \\ &= \sum_{q_1, q_2 \leq Q} q_1 q_2 \sum_{\substack{m', n', r' \\ q_1|m', q_2|n'}} c_{r'} a_{m'-r'} a_{n'+r'} \\ &= \sum_{q_1, q_2 \leq Q} q_1 q_2 \sum_{r' \in \mathbb{Z}} c_{r'} \sum_m a_{mq_1-r'} \sum_n a_{nq_2+r'}. \end{aligned}$$

The formula of the coefficients of the Fourier series of $G(t)$ gives

$$\begin{aligned} a_{mq_1-r} &= \int_{\mathbb{R}} G(t) e(-(mq_1-r)t) dt \\ &= \int_{\mathbb{R}} G(t) e(rt) e(-mq_1 t) dt \\ &= \int_{\mathbb{R}} \frac{1}{q_1} G\left(\frac{t}{q_1}\right) e\left(\frac{rt}{q_1}\right) e(-mt) dt. \end{aligned}$$

By the Poisson summation formula,

$$\sum_m a_{mq_1-r} = \sum_{m \in \mathbb{Z}} \frac{1}{q_1} G\left(\frac{m}{q_1}\right) e\left(\frac{rm}{q_1}\right).$$

Following a similar procedure, one obtains

$$\sum_m a_{nq_2+r} = \sum_{n \in \mathbb{Z}} \frac{1}{q_2} G\left(\frac{n}{q_2}\right) e\left(\frac{-rn}{q_2}\right).$$

Applying these two equations to the formula of S , we have

$$\begin{aligned} S &= \sum_{q_1, q_2 \leq Q} q_1 q_2 \sum_{r \in \mathbb{Z}} c_r \sum_{m \in \mathbb{Z}} \frac{1}{q_1} G\left(\frac{m}{q_1}\right) e\left(\frac{rm}{q_1}\right) \sum_{n \in \mathbb{Z}} \frac{1}{q_2} G\left(\frac{n}{q_2}\right) e\left(\frac{-rn}{q_2}\right) \\ &= \sum_{q_1, q_2 \leq Q} \sum_{m, n \in \mathbb{Z}} G\left(\frac{m}{q_1}\right) G\left(\frac{n}{q_2}\right) \sum_r c_r e\left(\left(\frac{m}{q_1} - \frac{n}{q_2}\right)r\right) \\ &= \sum_{q_1, q_2 \leq Q} \sum_{m, n \in \mathbb{Z}} G\left(\frac{m}{q_1}\right) G\left(\frac{n}{q_2}\right) \sum_r H\left(N\left(r + \frac{m}{q_1} - \frac{n}{q_2}\right)\right). \end{aligned}$$

Since $\text{Supp} G \subset (0, 1)$, $\text{Supp} H \subset (0, \Lambda)$ and $N \sim Q^2/2$, we have that if $r \neq 0$ then $G(m/q_1)G(n/q_2)H(N(r + m/q_1 - n/q_2)) = 0$ when N is large. Let $\delta = (q_1, q_2)$, $q_1 = d_1 \delta$ and $q_2 = d_2 \delta$. Let a, b satisfy $ad_1 + bd_2 = 1$ with $0 < b < d_1$. We consider a change in the index of summation given by $m = bm' + d_1 n'$ and $n = -am' + d_2 n'$. This yields

$$\begin{aligned} S &= \sum_{\substack{d_1 \delta, d_2 \delta \leq Q \\ (d_1, d_2)=1}} \sum_{m, n \in \mathbb{Z}} G\left(\frac{bm}{d_1 \delta} + \frac{n}{\delta}\right) G\left(\frac{-am}{d_2 \delta} + \frac{n}{\delta}\right) H\left(\frac{Nm}{d_1 d_2 \delta}\right) \\ &= \sum_{\substack{d_1 \delta, d_2 \delta \leq Q \\ (d_1, d_2)=1}} \sum_{m, n \in \mathbb{Z}} G\left(\frac{1}{\delta} \left(\frac{bm}{d_1} + n\right)\right) G\left(\frac{1}{\delta} \left(\frac{bm}{d_1} + n - \frac{m}{d_1 d_2}\right)\right) H\left(\frac{Nm}{d_1 d_2 \delta}\right). \end{aligned}$$

Since $\text{Supp} H \subset (0, \Lambda)$ and $N \geq (1 - \epsilon)Q^2/2$ for large Q , the summand is nonzero only if

$$m\delta \leq 2\Lambda/(1 - \epsilon).$$

By choosing ϵ sufficiently small, we have

$$m \leq 2\Lambda/\delta.$$

This inequality and the fact that $\text{Supp}G \subset (0, 1)$ imply that the summand is nonzero for finitely many values of n , which satisfy

$$0 < \frac{1}{\delta} \left(\frac{bm}{d_1} + n \right) < 1.$$

From the fact that

$$G \left(\frac{1}{\delta} \left(\frac{bm}{d_1} + n - \frac{m}{d_1 d_2} \right) \right) = G \left(\frac{1}{\delta} \left(\frac{bm}{d_1} + n \right) \right) + O \left(\frac{m}{\delta d_1 d_2} \right),$$

we have

$$\begin{aligned} S &= \sum_{\substack{d_1 \delta, d_2 \delta \leq Q \\ (d_1, d_2)=1}} \sum_{m \delta \leq 2\Lambda, n} G \left(\frac{1}{\delta} \left(\frac{bm}{d_1} + n \right) \right)^2 H \left(\frac{Nm}{d_1 d_2 \delta} \right) + O(\log^2 Q) \\ &= \sum_{m \delta \leq 2\Lambda, n} \sum_{\substack{d_1, d_2 \leq Q/\delta \\ (d_1, d_2)=1}} f_n \left(\frac{b}{d_1} \right) H \left(\frac{Nm}{d_1 d_2 \delta} \right) \end{aligned}$$

where

$$f_n(x) := G \left(\frac{1}{\delta} (mx + n) \right)^2.$$

Let K be a large positive integer whose value (as a function of Q) will be given later. The formula of S above becomes

$$\begin{aligned} S &= \sum_{\substack{m \delta \leq 2\Lambda \\ n \in \mathbb{Z}}} \sum_{i=1}^K \sum_{\substack{d_1, d_2 \leq Q/\delta \\ (d_1, d_2)=1 \\ b \in [d_1 i/K, d_1(i+1)/K]}} f_n \left(\frac{b}{d_1} \right) H \left(\frac{Nm}{d_1 d_2 \delta} \right) + O(\log^2 Q) \\ &= \sum_{\substack{m \delta \leq 2\Lambda \\ n \in \mathbb{Z}}} \sum_{i=1}^K \sum_{\substack{d_1, d_2 \leq Q/\delta \\ (d_1, d_2)=1 \\ b \in [d_1 i/K, d_1(i+1)/K]}} \left(f_n \left(\frac{i}{K} \right) + O \left(\frac{1}{K} \right) \right) H \left(\frac{Nm}{d_1 d_2 \delta} \right) + O(\log^2 Q) \\ &= \sum_{\substack{m \delta \leq 2\Lambda \\ n \in \mathbb{Z}}} \sum_{i=1}^K f_n \left(\frac{i}{K} \right) \sum_{\substack{d_1, d_2 \leq Q/\delta \\ (d_1, d_2)=1 \\ b \in [d_1 i/K, d_1(i+1)/K]}} H \left(\frac{Nm}{d_1 d_2 \delta} \right) + O \left(\frac{Q^2}{K} \right) + O(\log^2 Q). \end{aligned}$$

We note that if $\min(d_1, d_2) \leq Q^{1-\epsilon}$ then the main term is 0 for large Q . If $\min(d_1, d_2) > Q^{1-\epsilon}$ then the chain rule implies that $\|DH\|_\infty \ll 1/Q^{1-3\epsilon}$. Then Lemma 8 in [4] gives

$$\begin{aligned} \sum_{\substack{d_1, d_2 \leq Q/\delta \\ (d_1, d_2)=1 \\ b \in [d_1 i/K, d_1(i+1)/K]}} H \left(\frac{Nm}{d_1 d_2 \delta} \right) &= \frac{6}{\pi^2 K} \iint_{Q^{1-\epsilon} \leq x, y \leq Q/\delta} H \left(\frac{Nm}{xy\delta} \right) dx dy + O(Q^{3/2+\epsilon}) \\ &= \frac{6}{\pi^2 K} \iint_{Q^{1-\epsilon} \leq x, y \leq Q/\delta} H \left(\frac{Q^2 m}{2xy\delta} \right) dx dy + O(Q^{3/2+\epsilon}) \\ &= \frac{6Q^2}{\pi^2 K} \iint_{Q^{-\epsilon} \leq x, y \leq 1/\delta} H \left(\frac{m}{2xy\delta} \right) dx dy + O(Q^{3/2+\epsilon}). \end{aligned}$$

If $\min(x, y) \leq Q^{-\epsilon}$ then the integrand is 0 when Q is large. Thus the expression becomes

$$\frac{6Q^2}{\pi^2 K} \iint_{0 \leq x, y \leq 1/\delta} H\left(\frac{m}{2xy\delta}\right) dx dy + O(Q^{3/2+\epsilon}).$$

Putting this expression back into the formula of S , we have

$$\begin{aligned} S &= \frac{6Q^2}{\pi^2} \sum_{\substack{m\delta \leq 2\Lambda \\ n \in \mathbb{Z}}} \sum_{i=1}^K f_n\left(\frac{i}{K}\right) \frac{1}{K} \iint_{0 \leq x, y \leq 1/\delta} H\left(\frac{m}{2xy\delta}\right) dx dy + O(KQ^{3/2+\epsilon}) + O\left(\frac{Q^2}{K}\right) \\ &= \frac{6Q^2}{\pi^2} \sum_{\substack{m\delta \leq 2\Lambda \\ n \in \mathbb{Z}}} \int_0^1 f_n(x) dx \iint_{0 \leq x, y \leq 1/\delta} H\left(\frac{m}{2xy\delta}\right) dx dy + O(KQ^{3/2+\epsilon}) + O\left(\frac{Q^2}{K}\right). \end{aligned}$$

The definition of $f_n(x)$ yields

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \int_0^1 f_n(x) dx &= \sum_{n \in \mathbb{Z}} \int_0^1 G\left(\frac{1}{\delta}(mx + n)\right)^2 dx \\ &= \frac{\delta}{m} \sum_{n \in \mathbb{Z}} \int_{n/\delta}^{(m+n)/\delta} G(z)^2 dz \\ &= \delta \int_0^1 G(z)^2 dz. \end{aligned}$$

Choosing $K = [Q^{1/4}]$, we obtain

$$S = \frac{6Q^2}{\pi^2} \left(\int_0^1 G(z)^2 dz \right) \sum_{m\delta \leq 2\Lambda} \delta \int_{0 \leq x, y \leq 1/\delta} H\left(\frac{m}{2xy\delta}\right) dx dy + O(Q^{7/4+\epsilon}).$$

Let $\lambda = m/2\delta xy$. The main term is

$$\begin{aligned} &\frac{6Q^2}{\pi^2} \left(\int_0^1 G(z)^2 dz \right) \sum_{\delta \leq Q} \sum_{m \leq 2\Lambda/\delta} \delta \int_0^{1/\delta} \int_{m/2x}^{\Lambda} H(\lambda) \frac{m}{2\delta x \lambda^2} d\lambda dx \\ &= \frac{3Q^2}{\pi^2} \left(\int_0^1 G(z)^2 dz \right) \sum_{\delta \leq Q} \sum_{m\delta/2 \leq \Lambda} \int_{m\delta/2}^{\Lambda} \int_{m/2\lambda}^{1/\delta} \frac{H(\lambda)m}{x\lambda^2} dx d\lambda \\ &= \frac{3Q^2}{\pi^2} \left(\int_0^1 G(z)^2 dz \right) \sum_{\delta \leq Q} \sum_{m\delta/2 \leq \Lambda} \int_{m\delta/2}^{\Lambda} \frac{H(\lambda)m}{\lambda^2} \log \frac{2\lambda}{m\delta} d\lambda. \end{aligned}$$

Let $k = m\delta$. The expression becomes

$$\begin{aligned} &\frac{3Q^2}{\pi^2} \left(\int_0^1 G(z)^2 dz \right) \sum_{0 < k \leq 2\Lambda} \int_{k/2}^{\Lambda} \frac{H(\lambda)}{\lambda^2} \log \frac{2\lambda}{k} \sum_{m|k} m \\ &= \frac{3Q^2}{\pi^2} \left(\int_0^1 G(z)^2 dz \right) \int_0^{\Lambda} \frac{H(\lambda)}{\lambda^2} \sum_{1 \leq k \leq 2\lambda} \sigma(k) \log \frac{2\lambda}{k} d\lambda. \end{aligned}$$

We divide the expression by $N(b-a)$ and let $Q \rightarrow \infty$. The lemma follows after letting G and H approach the characteristic function of I and $(0, \Lambda)$ respectively.

We will apply the following theorem which is Theorem 2 from [2].

Theorem 4 Suppose $\mathcal{F} = (\mathcal{F}(Q))_{Q \in \mathbb{N}}$ is a sequence of finite sequences of points on a closed interval I with $\mathcal{F}(Q) = \{t_j^Q : 1 \leq j \leq N_Q\}$. Let $\mathcal{C} \subset \mathbb{R}^k$ be a curve with parametrization $f : I \rightarrow \mathcal{C}$, where the function f is continuous, piecewise continuously differentiable, and f' does not vanish in I . Denoting $x_j^Q = f(t_j^Q)$, we form a sequence of finite sequences of points on \mathcal{C} by letting $\mathcal{M}(Q) = \{x_j^Q : 1 \leq j \leq N_Q\}$ and $\mathcal{M} = \{\mathcal{M}(Q)\}_{Q \in \mathbb{N}}$. Suppose \mathcal{F} is uniformly distributed on I and for any subinterval I' of I , the sequence of functions $R_{\mathcal{F}_{I'}(Q)}(\lambda)$ converges pointwise as $Q \rightarrow \infty$ to a continuous function $R_{\mathcal{F}_{I'}}(\lambda)$ which is independent of the interval I' . Then the limiting pair correlation measure of \mathcal{M} on the curve \mathcal{C} exists and is given by

$$R_{\mathcal{M}_c}(\lambda) = \frac{1}{|I|} \int_I R_{\mathcal{F}_I} \left(\frac{l(\mathcal{C})}{|I||f'(t)|} \lambda \right) dt$$

where $l(\mathcal{C})$ is the length of the curve \mathcal{C} .

The theorem above implies that the limiting pair correlation measure of roots of $\prod_{k=1}^m H_k(z)$, counting multiplicities, exists on J as $m \rightarrow \infty$ and is given by

$$\begin{aligned} \frac{1}{|I|} \int_I R_{M_I} \left(\frac{l(J)}{|I||f'(t)|} \lambda \right) dt &= \frac{1}{|I|} \int_I \int_0^{l(J)\lambda/|I||f'(t)|} g_{M_I}(\lambda) d\lambda dt \\ &= \frac{l(J)}{|I|^2} \int_0^\lambda \int_I g_{M_I} \left(\frac{l(J)}{|I||f'(t)|} \lambda \right) \frac{dt}{|f'(t)|}. \end{aligned}$$

Hence Theorem 1 follows.

4 An explicit example of pair correlation on the corresponding curve

In this section we will consider the pair correlation for a specific sequence of polynomials. We have the following result.

Example Let $H_m(z)$ be a sequence of polynomials satisfying the generating function

$$\sum H_m(z)t^m = \frac{1}{z^2 t^2 + (z^2 - 2z)t + 1}.$$

Then the roots of $H_m(z)$ lie on the interval $[0, 4)$. The pair correlation function of the sequence of the roots of $\prod_{m=1}^n H_m(z)$, counting multiplicities, on the subinterval $J = [2 - 2\cos \pi a, 2 + 2\cos \pi a]$

with $0 < a < 1/2$ exists and is given by

$$\begin{aligned}
g_J(\lambda) = & \frac{6}{\pi \lambda^2 \cos \pi a} \sum_{\frac{4\lambda \cos \pi a}{\pi(1-2a)} < k \leq \frac{4\lambda \cos \pi a}{\pi(1-2a) \sin a}} \sigma(k) \left(-\sqrt{1 - \frac{16\lambda^2 \cos^2 \pi a}{\pi^2 k^2 (1-2a)^2}} + \log \left(1 + \sqrt{1 - \frac{16\lambda^2 \cos^2 \pi a}{\pi^2 k^2 (1-2a)^2}} \right) \right. \\
& \left. - \log \frac{4\lambda \cos \pi a}{\pi k (1-2a)} \right) + \frac{6}{\pi \lambda^2 \cos \pi a} \sum_{1 \leq k \leq \frac{4\lambda \cos \pi a}{\pi(1-2a) \sin a}} \sigma(k) \left(\left(1 + \log \frac{4\lambda \cos \pi a}{\pi k (1-2a)} \right) \cos \pi a \right. \\
& \left. - \frac{1}{2} \log \frac{1 + \cos \pi a}{1 - \cos \pi a} - \cos \pi a \log \sin \pi a \right).
\end{aligned}$$

To show this, let $z = x + iy$. From elementary computations, we note that

$$\begin{aligned}
\Im \frac{B^2(z)}{A(z)} &= \frac{2y(x^2 + y^2)P}{(x^2 + y^2)^2} \\
\Re \frac{B^2(z)}{A(z)} &= \frac{P^2 - Q^2}{(x^2 + y^2)^2}
\end{aligned}$$

where

$$\begin{aligned}
P &= 2x^2 + x^3 - 2y^2 + xy^2 \\
Q &= y(x^2 + y^2).
\end{aligned}$$

From Theorem 2, we have two cases, $y = 0$ or $P = 0$. The condition

$$\frac{P^2 - Q^2}{(x^2 + y^2)^2} \geq 0,$$

implies $y = 0$ in both cases. The inequality

$$\frac{P^2 - Q^2}{(x^2 + y^2)^2} \leq 4$$

gives $x^4(x^2 - 4x) \leq 0$. This implies that the roots of $H_m(z)$ lies on the real interval $[0, 4]$. The roots of $H_{30}(z)$ are given by Figure 3.

The definition of $h(z)$ in Theorem 1 gives

$$h(z) = \frac{B^2(z)}{A(z)} = (z - 2)^2.$$

The formula $f(t) := h^{-1}(4 \cos^2 \pi t)$ in Theorem 1 becomes

$$f(t) = 2 + 2 \cos \pi t.$$

This function maps the interval $I = (a, 1 - a)$ onto J with

$$f'(t) = -2\pi \sin \pi t.$$

Equation (2) gives

$$\begin{aligned}
g_J(\lambda) &= \frac{|J|}{|I|^2} \int_I g_I \left(\frac{|J|}{|I||f'(t)|} x \right) \frac{dt}{|f'(t)|} \\
&= \frac{24}{\pi|J|\lambda^2} \int_a^{1/2} \sum_{1 \leq k \leq |J|\lambda/\pi|I| \sin \pi t} \sigma(k) \sin \pi t \log \frac{|J|\lambda}{|I|k \sin \pi t} \\
&= \frac{24}{\pi|J|\lambda^2} \sum_{1 \leq k \leq |J|\lambda/\pi|I|} \sigma(k) \int_a^{1/2} \sin \pi t \log \frac{|J|\lambda}{|I|k \sin \pi t} dt \\
&\quad + \frac{24}{\pi|J|\lambda^2} \sum_{|J|\lambda/\pi I < k \leq |J|\lambda/\pi|I| \sin a} \sigma(k) \int_a^{t_k} \sin \pi t \log \frac{|J|\lambda}{|I|k \sin \pi t} dt
\end{aligned}$$

where $0 \leq t_k \leq 1/2$ is the solution to the equation

$$\sin \pi t_k = \frac{|J|\lambda}{|I|\pi k}.$$

The integrand in this formula has an antiderivative

$$-(1 + \log \frac{|J|\lambda}{\pi|I|k}) \cos \pi t + \frac{1}{2} \log(\cos \pi t + 1) - \frac{1}{2} \log(1 - \cos \pi t) + \cos \pi t \log \sin \pi t.$$

Thus the pair correlation function $g_J(\lambda)$ becomes

$$\begin{aligned}
&\frac{24}{\pi|J|\lambda^2} \sum_{|J|\lambda/\pi I < k \leq |J|\lambda/\pi|I| \sin a} \sigma(k) \left(-\sqrt{1 - \frac{|J|^2 \lambda^2}{\pi^2 |I|^2 k^2}} + \log \left(1 + \sqrt{1 - \frac{|J|^2 \lambda^2}{\pi^2 |I|^2 k^2}} \right) - \log \frac{|J|\lambda}{|I|\pi k} \right) \\
&\quad + \frac{24}{\pi|J|\lambda^2} \sum_{1 \leq k \leq |J|\lambda/\pi|I| \sin a} \sigma(k) \left((1 + \log \frac{|J|\lambda}{\pi|I|k}) \cos \pi a - \frac{1}{2} \log \frac{1 + \cos \pi a}{1 - \cos \pi a} - \cos \pi a \log \sin \pi a \right).
\end{aligned}$$

The corollary follows by noticing that $|J| = 4 \cos \pi a$ and $|I| = 1 - 2a$.

We notice that when a approaches 0, the series in the formula of $g_J(\lambda)$ behaves similar to

$$\sum_{k=1}^{\infty} \frac{\sigma(k)}{k^2}$$

which diverges. Hence the limiting pair correlation function does not exist on the whole interval $[0, 4]$.

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